Some Functorial Properties of Nilpotent Multipliers

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Abstract

In this paper, we are going to look at the c-nilpotent multiplier of a group G, $\mathcal{N}_cM(G)$, as a functor from the category of all groups, $\mathcal{G}roup$, to the category of all abelian groups, $\mathcal{A}b$, and focusing on some functional properties of it. In fact, by using some results of the first author and others and finding an explicit formula for the c-nilpotent multiplier of a finitely generated abelian group, we try to concentrate on the commutativity of the above functor with the two famous functors Ext and Tor.

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1. Introduction

Let $G \cong F/R$ be a group, presented as a quotient group of a free group F by a normal subgroup R. Then the *Baer-invariant* of G, after R. Baer [1], with respect to the variety $\mathcal{V}$, denoted by $\mathcal{V}M(G)$, is defined to be

$$\mathcal{V}M(G) = \frac{R \cap V(F)}{[RV^*F]} \ ,$$

where $V(F)$ is the verbal subgroup of F with respect to $\mathcal{V}$ and

$$[RV^*F] =< v(f_1, \ldots, f_{i-1}, f_i r, f_{i+1}, \ldots, f_n) v(f_1, \ldots, f_i, \ldots f_n)^{-1} \mid r \in R,$$

$$1 \le i \le n, v \in V, f_i \in F, n \in \mathbf{N} > .$$

It can be proved that the Baer-invariant of a group G is independent of the choice of the presentation of G and it is always an abelian group (See [8]).

In particular, if $\mathcal{V}$ is the variety of abelian groups, $\mathcal{A}$, then the Baer-invariant of G will be $(R \cap F')/[R, F]$, which, following Hopf [6], is isomorphic to the second

cohomology group of G, $H_2(G, \mathbf{C}^*)$, in finite case, and also is isomorphic to the well-known notion the *Schur multiplier* of G, denoted by $M(G)$. The multiplier $M(G)$ arose in Schur's work [15] of 1904 on projective representations of a group, and has subsequently found a variety of other applications. The survey article of Wiegold [19] and the books by Beyl and Tappe [2] and Karpilovsky [7] form a fairly comprehensive account of $M(G)$.

If $\mathcal{V}$ is the variety of nilpotent groups of class at most $c \geq 1$, $\mathcal{N}_c$, then the Baer-invariant of the group G will be

$$\mathcal{N}_c M(G) = \frac{R \cap \gamma_{c+1}(F)}{[R,\ {}_cF]} \ ,$$

where $\gamma_{c+1}(F)$ is the $(c+1)$st term of the lower central series of F and $[R,_1 F] = [R, F]$, $[R,_c F] = [[R,_{c-1} F], F]$, inductively. The above notion is also called the c-nilpotent multiplier of G and denoted by $M^{(c)}(G)$ (see [3]).

The following theorem permit us to look at the notion of the Baer-invariant as a functor.

Theorem 1.1.

Let $\mathcal{V}$ be an arbitrary variety of groups. Then, using the notion of the Baer-invariant, we can consider the following covariant functor from the category of all groups, $\mathcal{G}roup$, to the category of all abelian groups, $\mathcal{A}b$

$$\mathcal{V}M(-)\ :\ \mathcal{G}roup \longrightarrow \mathcal{A}b\ ,$$

which assigns to any group G the abelian group $\mathcal{V}M(G)$.

Proof. Let G be an arbitrary group. By the properties of the Baer-invariant, $\mathcal{V}M(G)$ is independent of the choice of a presentation of G and it is always abelian. So $\mathcal{V}M(-)$ assigns an abelian group to each group G. Also, if G_1 and G_2 are two arbitrary groups with the following presentations:

$$1 \longrightarrow R_1 \longrightarrow F_1 \stackrel{\pi_1}{\longrightarrow} G_1 \longrightarrow 1 \quad , \quad 1 \longrightarrow R_2 \longrightarrow F_2 \stackrel{\pi_2}{\longrightarrow} G_2 \longrightarrow 1\ ,$$

and if $\phi : G_1 \to G_2$ is a homomorphism, then, using the universal property of free groups, there exists a homomorphism $\overline{\phi} : F_1 \to F_2$. It is easy to see that $\overline{\phi}$ induces a homomorphism

$$\hat{\phi} : \frac{R_1 \cap V(F_1)}{[R_1 V^* F_1]} \longrightarrow \frac{R_2 \cap V(F_2)}{[R_2 V^* F_2]}\ ,$$

i.e. $\overline{\phi}\ :\ \mathcal{V}M(G_1) \longrightarrow \mathcal{V}M(G_2)$ is a homomorphism from the Baer-invariant of G_1 to the Baer-invariant of G_2. It is a routine verification to see that the above assignment is a functor from $\mathcal{G}roup$ to $\mathcal{A}b$ (see also [8]).$\square$

§2. Elementary Results

Being additive is usually one of the important property that a functor may have. Unfortunately, the c-nilpotent multiplier functor $\mathcal{N}_cM(-)$ is *not* additive even if we restrict ourself to abelian groups. The following theorems can prove this claim.

Theorem 2.1 (I. Schur [14], J. Wiegold [16]).

Let $G = A \times B$ be the direct product of two groups A and B. Then

$$M(G) \cong M(A) \oplus M(B) \oplus (A_{ab} \otimes B_{ab}) .$$

Theorem 2.2 (B. Mashayekhy and M.R.R. Moghaddam [11]).

Let $G \cong \mathbf{Z}_{n_1} \oplus \mathbf{Z}_{n_2} \oplus \ldots \oplus \mathbf{Z}_{n_k}$, be a finite abelian group, where $n_{i+1}|n_i$ for all $1 \le i \le k-1$ and $k \ge 2$. Then, for all $c \ge 1$, the c-nilpotent multiplier of G is

$$\mathcal{N}_cM(G) \cong \mathbf{Z}_{n_2}^{(b_2)} \oplus \mathbf{Z}_{n_3}^{(b_3-b_2)} \oplus \ldots \oplus \mathbf{Z}_{n_k}^{(b_k-b_{k-1})} \quad ,$$

where $\mathbf{Z}_m^{(n)}$ denotes the direct sum of n copies of the cyclic group $\mathbf{Z}_m$, and b_i is the number of basic commutators of weight $c+1$ on i letters (see [5]).

One of the interesting corollary of Theorem 2.2 is that the c-nilpotent multiplier functors can preserve every elementary abelian p-group.

Corollary 2.3.

Let $G = \mathbf{Z}_p \oplus \ldots \oplus \mathbf{Z}_p$ (k-copies) be an elementary abelian p-group. Then, for all $c \ge 1$, $\mathcal{N}_cM(G)$ is also an elementary abelian p-group.

Proof. By Theorem 2.2 we have

$$\mathcal{N}_cM(G) \cong \mathbf{Z}_p^{(b_2)} \oplus \mathbf{Z}_p^{(b_3-b_2)} \oplus \ldots \oplus \mathbf{Z}_p^{(b_k-b_{k-1})} = \mathbf{Z}_p \oplus \ldots \oplus \mathbf{Z}_p \ \ (b_k - copies).$$

Hence the result holds. Note that $|G| = p^n$ and $|\mathcal{N}_cM(G)| = p^{b_k}$.□

In 1952, C. Miller [12] proved that the Schur multiplier of a free product is isomorphic to the direct sum of the Schur multipliers of the free factors. In other words, he proved that the Schur multiplier functor $M(-)$ *is coproduct-preserving.*

Theorem 2.4 (C. Miller [12]).

For any group G_1 and G_2,

$$M(G_1 * G_2) \cong M(G_1) \oplus M(G_2) ,$$

where $G_1 * G_2$ is the free product of G_1 and G_2.

Now, with regards to the above theorem, it seems natural to ask whether the c-nilpotent multiplier functors, $\mathcal{N}_cM(-)$, $c \ge 2$, are coproduct-preserving or not. To answer the question, first we state an important theorem of J. Burns and G. Ellis [3, Proposition 2.13 and its Erratum] which is proved by a homological method.

Theorem 2.5 (J. Burns and G. Ellis [3]).

Let G and H be two arbitrary groups, then there is an isomorphism

$$\mathcal{N}_2M(G * H) \cong$$

$$\mathcal{N}_2M(G)\oplus\mathcal{N}_2M(H)\oplus(M(G)\otimes H_{ab})\oplus(G_{ab}\otimes M(H))\oplus Tor_1^{\mathbf{Z}}(G_{ab},H_{ab})\ .$$

Now, using the above theorem and properties of tensor product and $Tor_1^{\mathbf{Z}}$, we can prove that the second nilpotent multiplier functor $\mathcal{N}_2M(-)$, preserves the coproduct of a finite family of cyclic groups of mutually coprime order.

Corollary 2.6.

Let $\{\mathbf{Z}_{n_i}|1\le i\le m\}$ be a family of cyclic groups of mutually coprime order. Then

$$\mathcal{N}_2M(\prod_{i=1}^{m}{}^*\mathbf{Z}_{n_i})\cong\oplus\sum_{i=1}^{m}\mathcal{N}_2M(\mathbf{Z}_{n_i})\ ,$$

where $\prod_{i=1}^{m}{}^*\mathbf{Z}_{n_i}$ is the free product of $\mathbf{Z}_{n_i}$'s, $1\le i\le n$.

Proof. By using induction on m and the following properties the result holds.

$$\mathcal{N}_2M(\mathbf{Z}_{n_i})\cong 1, Tor_1^{\mathbf{Z}}(\mathbf{Z}_{n_i},\mathbf{Z}_{n_j})\cong\mathbf{Z}_{n_i}\otimes\mathbf{Z}_{n_j}=1,\ for\ all\ i\neq j.\square$$

Note that the first author has generalized the above corollary to the variety of nilpotent groups of class at most $c,\mathcal{N}_c$, for all $c\ge 2$ as follows.

Theorem 2.7 (B. Mashayekhy [10]).

Let $\{\mathbf{Z}_{n_i}|1\le i\le m\}$ be a family of cyclic groups of mutually coprime order. Then

$$\mathcal{N}_cM(\prod_{i=1}^{m}{}^*\mathbf{Z}_{n_i})\cong\oplus\sum_{i=1}^{m}\mathcal{N}_cM(\mathbf{Z}_{n_i})\ ,\ for\ all\ c\ge 1.$$

In the following example, we are going to show that the condition of being mutually coprime order for the family of cyclic groups $\{\mathbf{Z}_{n_i}|1\le i\le m\}$ is very essential in the above results. In other words, we show that the second nilpotent multiplier functor, $\mathcal{N}_2M(-)$, *is not coproduct preserving*, in general.

Example.

Let $D_\infty=<a,b|a^2=b^2=1>\cong\mathbf{Z}_2*\mathbf{Z}_2$ be the infinite dihedral group. Then

$$\mathcal{N}_2M(D_\infty)\ncong\mathcal{N}_2M(\mathbf{Z}_2)\oplus\mathcal{N}_2M(\mathbf{Z}_2)\quad .$$

Proof. By Theorem 2.5 we have

$$\begin{aligned}\mathcal{N}_2M(D_\infty)&\cong\mathcal{N}_2M(\mathbf{Z}_2)\oplus\mathcal{N}_2M(\mathbf{Z}_2)\oplus\mathbf{Z}_2\otimes M(\mathbf{Z}_2)\oplus M(\mathbf{Z}_2)\otimes\mathbf{Z}_2\oplus Tor_1^{\mathbf{Z}}(\mathbf{Z}_2,\mathbf{Z}_2)\\&\cong Tor_1^{\mathbf{Z}}(\mathbf{Z}_2,\mathbf{Z}_2)\cong\mathbf{Z}_2\otimes\mathbf{Z}_2\cong\mathbf{Z}_2\ .\end{aligned}$$

But $\mathcal{N}_2M(\mathbf{Z}_2)\oplus\mathcal{N}_2M(\mathbf{Z}_2)=1$. Hence the result holds. $\square$

Note.

In 1980 M.R.R. Moghaddam [12] proved that in general, the Baer-invariant functor commutes with direct limit of a directed system of groups.

We know that every functor can preserve any split exact sequence as a split sequence. This property gives us the following interesting result.

Theorem 2.8.

Let $G = T \overset{\theta}{\rtimes} N$ be the semidirect product (splitting extension) of N by T under θ. Then $\mathcal{V}M(T)$ is a direct summand of $\mathcal{V}M(G)$, for every variety of groups $\mathcal{V}$.

Note that K.I. Tahara [15] 1972, and W. Haebich [4] 1977, tried to obtain a result similar to the above theorem for the Schur multiplier of a semidirect product with an emphasis on finding the structure of the complementary factor $M(T)$ of $M(G)$, as much as possible. Also, a generalization of Haebich's result [4] presented by the first author in [9].

Finally, the properties of right and left exactness are some of the most interesting properties that a functor may have. In the following, we show that the c-nilpotent multiplier functors *are not right or left exact.*

Theorem 2.9.

For every $c \geq 1$, the c-nilpotent multiplier functor, $\mathcal{N}_cM(-)$, is not right exact.

Proof. Let G be a group such that $\mathcal{N}_cM(G) \neq 1$ (note that by Theorem 2.2, we can always find such a group G). Let F be a free group and $\pi : F \to G$ be an epimorphism (we can always consider a free presentation for a group G). Now by definition of the Baer-invariant we have $\mathcal{N}_cM(F) = 1$ (consider the free presentation $1 \to 1 \to F \to F \to 1$ for F). Therefore, it is easy to see that $\mathcal{N}_cM(F) \longrightarrow \mathcal{N}_cM(G)$ is not onto. $\square$

Theorem 2.10.

The c-nilpotent multiplier functor, $\mathcal{N}_cM(-)$, is not left exact, in general.

Proof. Suppose $G = \mathbf{Z}_4 \oplus \mathbf{Z}_4$. Then by Theorem 2.1 we have

$$M(G) \cong M(\mathbf{Z}_4) \oplus M(\mathbf{Z}_4) \oplus (\mathbf{Z}_4 \otimes \mathbf{Z}_4) \cong \mathbf{Z}_4 \ .$$

By a famous result on the Schur multiplier we know that every finite p-group can be embedded in a finite p-group whose Schur multiplier is elementary abelian p-group (see [7,17]). So there exists an exact sequence $G \overset{\theta}{\to} H \to 1$, where H is a finite 2-group and $M(H)$ is an elementary abelian 2-group. Hence $M(\theta) : M(G) \to M(H)$ can not be a monomorphism. $\square$

3. Main Results

In this section, we will see the behaviour of the functor $\mathcal{N}_cM(-)$ with the functors $Ext^n_{\mathbf{Z}}(\mathbf{Z_m}, -)$ and $Tor^{\mathbf{Z}}_n(\mathbf{Z_m}, -)$. First, by using notations and similar method of paper [11], we can present an explicit formula for the c-nilpotent multiplier of a finitely generated abelian groups as follows.

Theorem 3.1.

Let $G \cong \mathbf{Z}^{(n)} \oplus \mathbf{Z}_{n_1} \oplus \mathbf{Z}_{n_2} \oplus \ldots \oplus \mathbf{Z}_{n_k}$, be a finitely generated abelian group, where $n \geq 0$, $n_{i+1}|n_i$ for all $1 \leq i \leq k-1$ and $k \geq 2$. Then, for all $c \geq 1$, the

c-nilpotent multiplier of G is

$$\mathcal{N}_cM(G) \cong \mathbf{Z}^{(b_n)} \oplus \mathbf{Z}_{n_1}^{(b_{n+1}-b_n)} \oplus \ldots \oplus \mathbf{Z}_{n_k}^{(b_{n+k}-b_{n+k-1})} \quad ,$$

where $b_1 = b_0 = 0$
Proof. Clearly $\mathbf{Z} \otimes \mathbf{Z} \cong \mathbf{Z}$, $\mathbf{Z} \otimes \mathbf{Z}_{n_i} \cong \mathbf{Z}_{n_i}$ and $\mathbf{Z}_{n_i} \otimes \mathbf{Z}_{n_{i+1}} \cong \mathbf{Z}_{n_{i+1}}$. Hence we have

$$\mathbf{Z}^{(t)} \otimes \mathbf{Z}_{n_1} \otimes \mathbf{Z}_{n_2} \otimes \ldots \otimes \mathbf{Z}_{n_r} \overset{(*)}{\cong} \mathbf{Z}_{n_r} \quad \text{and} \quad \mathbf{Z} \otimes \ldots \otimes \mathbf{Z} \cong \mathbf{Z}.$$

for all $t \geq 0$ and $r \geq 1$. Thus by theorem 2.3 of [11] we have

$$\mathcal{N}_cM(\mathbf{Z}^{(n)}) \cong T(\mathbf{Z}, \ldots, \mathbf{Z})_{c+1} \cong \mathbf{Z}^{(b_n)}.$$

We remind that $T(H_1, \ldots, H_n)_{c+1}$ is the summation of all the tensor products corresponding to the subgroup generated by all the basic commutators of weight $c+1$ on n letters $x_1, \ldots, x_n$, where $x_i \in H_i$ for all $1 \leq i \leq n$. Now, by induction hypothesis assume

$$\mathcal{N}_cM(\mathbf{Z}^{(n)} \oplus \mathbf{Z}_{n_1} \oplus \mathbf{Z}_{n_2} \oplus \ldots \oplus \mathbf{Z}_{n_{k-1}}) \cong \mathbf{Z}^{(b_n)} \oplus \mathbf{Z}_{n_1}^{(b_{n+1}-b_n)} \oplus \ldots \oplus \mathbf{Z}_{n_{k-1}}^{(b_{n+k-1}-b_{n+k-2})} \ .$$

Then we have

$$\mathcal{N}_cM(\mathbf{Z}^{(n)} \oplus \mathbf{Z}_{n_1} \oplus \mathbf{Z}_{n_2} \oplus \ldots \oplus \mathbf{Z}_{n_k}) \cong T(\underbrace{\mathbf{Z}, \ldots, \mathbf{Z}}_{n-times}, \mathbf{Z}_{n_1}, \ldots, \mathbf{Z}_{n_k})_{c+1}$$

$$\cong T(\underbrace{\mathbf{Z}, \ldots, \mathbf{Z}}_{n-times}, \mathbf{Z}_{n_1}, \ldots, \mathbf{Z}_{n_{k-1}})_{c+1} \oplus L \ ,$$

where L is the summation of all the tensor products of $\mathbf{Z}, \mathbf{Z}_{n_1}, \ldots, \mathbf{Z}_{n_k}$ corresponding to the subgroup generated by all the basic commutators of weight $c+1$ on $n+k$ letters which involve $\mathbf{Z}_{n_k}$. Using $(*)$, all those tensor product are isomorphic to $\mathbf{Z}_{n_k}$. So L is the direct summand of $(b_{n+k} - b_{n+k-1})$-copies of $\mathbf{Z}_{n_k}$. Hence the result follows by induction. □

For the rest of the paper we need the following lemmas.

Lemma 3.2.

For any abelian groups A and B, we have
(i) $Ext^1_{\mathbf{Z}}(\mathbf{Z}/m\mathbf{Z}, B) \cong B/mB$.
Also, $Ext^n_{\mathbf{Z}}(A, B) = 0$, for all $n \geq 2$.
(ii) If A and B are finite abelian groups, then

$$Ext^1_{\mathbf{Z}}(A, B) \cong Ext^1_{\mathbf{Z}}(B, A).$$

(iii) $Tor^{\mathbf{Z}}_1(\mathbf{Z}/m\mathbf{Z}, B) \cong B[m]$, where $B[m] = \{b \in B : mb = 0\}$. Also, $Tor^{\mathbf{Z}}_n(A, B) = 0$, for all $n \geq 2$, and $Tor^{\mathbf{Z}}_1(A, B) \cong Tor^{\mathbf{Z}}_1(B, A)$.

Proof. See [14, Chapters 7, 8]. □

Lemma 3.3.

Let A and $\{B_k\}_{k\in I}$ be abelian groups. Then for all $n\geq 0$ the following isomorphism hold.

$$(i)\ Ext^n_{\mathbf{Z}}(A,\prod_{k\in I}B_k)\cong\prod_{k\in I}Ext^n_{\mathbf{Z}}(A,B_k),\ Ext^n_{\mathbf{Z}}(\coprod_{k\in I}B_k,A)\cong\prod_{k\in I}Ext^n_{\mathbf{Z}}(B_k,A).$$

$$(ii)\ Tor^{\mathbf{Z}}_n(A,\coprod_{k\in I}B_k)\cong\coprod_{k\in I}Tor^{\mathbf{Z}}_n(A,B_k),\ Tor^{\mathbf{Z}}_n(\coprod_{k\in I}B_k,A)\cong\coprod_{k\in I}Tor^{\mathbf{Z}}_n(B_k,A).$$

Proof. See [14]. $\square$

It is obvious that the functor $\mathcal{N}_cM(-)$ commutes with the functors $Ext^n_{\mathbf{Z}}(\mathbf{Z}_m,-)$, and $Tor^{\mathbf{Z}}_n(\mathbf{Z}_m,-)$ for all $n\geq 2$, by lemma 3.2. Now we are going to pay our attention to the functors $Ext^1_{\mathbf{Z}}(\mathbf{Z}_m,-)$, $Ext^1_{\mathbf{Z}}(-,\mathbf{Z}_m)$, and $Tor^{\mathbf{Z}}_1(\mathbf{Z}_m,-)$.

Theorem 3.4.

Let $D\cong\mathbf{Z}^{(n)}\oplus\mathbf{Z}_{n_1}\oplus\mathbf{Z}_{n_2}\oplus\ldots\oplus\mathbf{Z}_{n_k}$, be a finitely generated abelian group, where $n\geq 0$, $n_{i+1}|n_i$ for all $1\leq i\leq k-1$. Then, for all $c\geq 1$, the following isomorphisms hold.

(i) $\mathcal{N}_cM\big(Ext^1_{\mathbf{Z}}(\mathbf{Z}_m,D)\big)\cong\mathbf{Z}_m^{(b_n)}\oplus(\oplus\sum_{i=1}^k\mathbf{Z}_{(n_i,m)}^{(b_{n+i}-b_{n+i-1})})$.

(ii) $Ext^1_{\mathbf{Z}}(\mathbf{Z}_m,\mathcal{N}_cM(D))\cong\mathbf{Z}_m^{(b_n)}\oplus(\oplus\sum_{i=1}^k\mathbf{Z}_{(n_i,m)}^{(b_{n+i}-b_{n+i-1})})$.

(iii) $\mathcal{N}_cM(Ext^1_{\mathbf{Z}}(D,\mathbf{Z}_m))\cong\oplus\sum_{i=2}^k\mathbf{Z}_{(n_i,m)}^{(b_i-b_{i-1})}$.

(iv) $Ext^1_{\mathbf{Z}}(\mathcal{N}_cM(D),\mathbf{Z}_m)\cong\oplus\sum_{i=1}^k\mathbf{Z}_{(n_i,m)}^{(b_{n+i}-b_{n+i-1})}$.

(v) $\mathcal{N}_cM(Tor^{\mathbf{Z}}_1(\mathbf{Z}_m,D))\cong\oplus\sum_{i=2}^k\mathbf{Z}_{(n_i,m)}^{(b_i-b_{i-1})}$.

(vi) $Tor^{\mathbf{Z}}_1(\mathbf{Z}_m,\mathcal{N}_cM(D))\cong\oplus\sum_{i=1}^k\mathbf{Z}_{(n_i,m)}^{(b_{n+i}-b_{n+i-1})}$.

Proof. (i) By Lemma 3.3(i), $Ext^1_{\mathbf{Z}}(\mathbf{Z}/m\mathbf{Z},\mathbf{Z})\cong\mathbf{Z}/m\mathbf{Z}\cong\mathbf{Z}_m$. Now by using Lemmas 3.3(i) and 3.2(i), we have

$$Ext^1_{\mathbf{Z}}(\mathbf{Z}_m,D)\cong(Ext^1_{\mathbf{Z}}(\mathbf{Z}_m,\mathbf{Z}))^{(n)}\oplus(\oplus\sum_{i=1}^k Ext^1_{\mathbf{Z}}(\mathbf{Z}_m,\mathbf{Z}_{n_i}))$$

$$\cong\mathbf{Z}_m^{(n)}\oplus(\oplus\sum_{i=1}^k\mathbf{Z}_{n_i}/m\mathbf{Z}_{n_i}).$$

One can see that for every $n,m\in\mathbf{Z}$, we have $\mathbf{Z}_m/n\mathbf{Z}_m\cong\mathbf{Z}_{(n,m)}$. Therefore

$$Ext^1_{\mathbf{Z}}(\mathbf{Z}_m,D)\cong\mathbf{Z}_m^{(n)}\oplus(\oplus\sum_{i=1}^k\mathbf{Z}_{(n_i,m)}).$$

Now, by Theorem 2.2 and by noting that $(m,n_{i+1})|(m,n_i)|m$ we have

$$\mathcal{N}_cM(Ext^1_{\mathbf{Z}}(\mathbf{Z}_m,D))$$

$$\cong \mathbf{Z}_m^{(b_2-b_1)} \oplus \mathbf{Z}_m^{(b_3-b_2)} \oplus \ldots \oplus \mathbf{Z}_m^{(b_n-b_{n-1})} \oplus \mathbf{Z}_{(n_1,m)}^{(b_{n+1}-b_n)} \oplus \ldots \oplus \mathbf{Z}_{(n_k,m)}^{(b_{n+k}-b_{n+k-1})}$$

$$\cong \mathbf{Z}_m^{(b_n)} \oplus (\oplus \sum_{i=1}^{k} \mathbf{Z}_{(n_i,m)}^{(b_{n+i}-b_{n+i-1})}).$$

(ii) By Theorem 3.1 and Lemmas 3.3(i) and 3.2(i), we have

$$\begin{aligned} Ext^1_{\mathbf{Z}}(\mathbf{Z}_m, \mathcal{N}_cM(D)) &\cong Ext^1_{\mathbf{Z}}(\mathbf{Z}_m, \mathbf{Z})^{(b_n)} \oplus (\oplus \textstyle\sum_{i=1}^k (Ext^1_{\mathbf{Z}}(\mathbf{Z}_m, \mathbf{Z}_{n_i}))^{(b_{n+i}-b_{n+i-1})}) \\ &\cong \mathbf{Z}_m^{(b_n)} \oplus (\oplus \textstyle\sum_{i=1}^k \mathbf{Z}_{(n_i,m)}^{(b_{n+i}-b_{n+i-1})}). \end{aligned}$$

(iii) By Lemmas 3.3(ii) and 3.2(ii) we have

$$Tor_1^{\mathbf{Z}}(\mathbf{Z}_m, D) \cong (Tor_1^{\mathbf{Z}}(\mathbf{Z}_m, \mathbf{Z}))^{(n)} \oplus (\oplus \sum_{i=1}^{k} Tor_1^{\mathbf{Z}}(\mathbf{Z}_m, \mathbf{Z}_{n_i})) \cong \oplus \sum_{i=1}^{k} \mathbf{Z}_{n_i}[m].$$

Note that $Tor_1^{\mathbf{Z}}(\mathbf{Z}_m, \mathbf{Z}) \cong 1$ and $\mathbf{Z}_n[m] \cong \mathbf{Z}_{(m,n)}$. So we have $Tor_1^{\mathbf{Z}}(\mathbf{Z}_m, D) \cong \oplus \sum_{i=1}^k \mathbf{Z}_{(n_i,m)}$. Now by Theorem 2.2 the result holds.
(iv) Again by using Theorem 3.1 and Lemmas 3.3(ii) and 3.2(ii), we have

$$Tor_1^{\mathbf{Z}}(\mathbf{Z}_m, \mathcal{N}_cM(D)) \cong (Tor_1^{\mathbf{Z}}(\mathbf{Z}_m, \mathbf{Z}))^{(b_n)} \oplus (\oplus \sum_{i=1}^{k} Tor_1^{\mathbf{Z}}(\mathbf{Z}_m, \mathbf{Z}_{n_i}^{(b_{n+i}-b_{n+i-1})})$$

$$\cong \oplus \sum_{i=1}^{k} Tor_1^{\mathbf{Z}}(\mathbf{Z}_m, \mathbf{Z}_{n_i}^{(b_{n+i}-b_{n+i-1})}) \cong \oplus \sum_{i=1}^{k} \mathbf{Z}_{(n_i,m)}^{(b_{n+i}-b_{n+i-1})}. \square$$

In the following corollary you can find some of main results of the paper.
Corollary 3.5.

Let D be an arbitrary finitely generated abelian group. Then
(i) $\mathcal{N}_cM\big(Ext^1_{\mathbf{Z}}(\mathbf{Z}_m, D)\big) \cong Ext^1_{\mathbf{Z}}(\mathbf{Z}_m, \mathcal{N}_cM(D))$.
(ii) If D is also finite, then

$$\mathcal{N}_cM(Tor_1^{\mathbf{Z}}(\mathbf{Z}_m, D)) \cong Tor_1^{\mathbf{Z}}(\mathbf{Z}_m, \mathcal{N}_cM(D)),$$

$$\mathcal{N}_cM(Ext^1_{\mathbf{Z}}(D, \mathbf{Z}_m)) \cong Ext^1_{\mathbf{Z}}(\mathcal{N}_cM(D), \mathbf{Z}_m).$$

(iii) If D is infinite, then

$$\mathcal{N}_cM(Tor_1^{\mathbf{Z}}(\mathbf{Z}_m, D)) \ncong Tor_1^{\mathbf{Z}}(\mathbf{Z}_m, \mathcal{N}_cM(D)).$$

$$\mathcal{N}_cM(Ext^1_{\mathbf{Z}}(D, \mathbf{Z}_m)) \ncong Ext^1_{\mathbf{Z}}(\mathcal{N}_cM(D), \mathbf{Z}_m).$$

This means that the c-nilpotent multiplier functors, $\mathcal{N}_cM(-)$ *do not* commute with $Tor_1^{\mathbf{Z}}(\mathbf{Z}_m,-)$ and $Ext^1_{\mathbf{Z}}(-,\mathbf{Z}_m)$, in infinite case.

Proof. (i) It is clear by parts (i), (ii) of the previous theorem.
(ii) By putting $n=0$ in parts (iii) to (vi) of the previous theorem, the result holds.
(iii) Since D is infinite, so $n\geq 1$. Hence the result holds by the previous theorem parts (iii) to (vi). □

We know that $Hom(\mathbf{Z}_m,\mathbf{Z})\cong 0$ and $Hom(\mathbf{Z},\mathbf{Z}_m)\cong \mathbf{Z}_m$. So by similar methods of Theorem 3.4 we are going to indicate the behaviour of functor $\mathcal{N}_cM(-)$ with $Ext^0_{\mathbf{Z}}(\mathbf{Z}_m,-)=Hom(\mathbf{Z}-m,-)$, $Ext^0_{\mathbf{Z}}(-,\mathbf{Z}_m)=Hom(-,\mathbf{Z}_m)$, and $Tor_0^{\mathbf{Z}}(\mathbf{Z}_m,-)=\mathbf{Z}_m\otimes -$ as the following theorem.

Theorem 3.6.

For any finitely generated abelian group $D\cong \mathbf{Z}^{(n)}\oplus\mathbf{Z}_{n_1}\oplus\mathbf{Z}_{n_2}\oplus\ldots\oplus\mathbf{Z}_{n_k}$, we have

(i) $\mathcal{N}_cM(Hom(\mathbf{Z}_m,D)\cong \mathbf{Z}^{(b_2)}_{(m,n_2)}\oplus\ldots\oplus\mathbf{Z}^{(b_k-b_{k-1})}_{(m,n_k)}$.
(ii) $Hom(\mathbf{Z}_m,\mathcal{N}_cM(D))\cong \mathbf{Z}^{(b_{n+1}-b_n)}_{(m,n_1)}\oplus\ldots\oplus\mathbf{Z}^{(b_{n+k}-b_{n+k-1})}_{(m,n_k)}$.
(iii) If D is finite, then $\mathcal{N}_cM(Hom(\mathbf{Z}_m,D)\cong Hom(\mathbf{Z}_m,\mathcal{N}_cM(D))$.
If D is infinite, then $\mathcal{N}_cM(Hom(\mathbf{Z}_m,D)\ncong Hom(\mathbf{Z}_m,\mathcal{N}_cM(D))$.
(iv) $\mathcal{N}_cM(Hom(D,\mathbf{Z}_m))\cong Hom(\mathcal{N}_cM(D),\mathbf{Z}_m)\cong \mathbf{Z}_m^{(b_n)}\oplus\mathbf{Z}^{(b_{n+1}-b_n)}_{(m,n_1)}\oplus\ldots\oplus\mathbf{Z}^{(b_{n+k}-b_{n+k-1})}_{(m,n_k)}$.
(v) $\mathcal{N}_cM(\mathbf{Z}_m\otimes D)\cong\mathbf{Z}_m\otimes\mathcal{N}_cM(D)\cong \mathbf{Z}_m^{(b_n)}\oplus\mathbf{Z}^{(b_{n+1}-b_n)}_{(m,n_1)}\oplus\ldots\oplus\mathbf{Z}^{(b_{n+k}-b_{n+k-1})}_{(m,n_k)}$.

Now, in the following we are going to show that our conditions in the previous results are essential. In general case $Ext^i_{\mathbf{Z}}(A,-)$ and $Tor_i^{\mathbf{Z}}(A,-)$, where A is not cyclic, *do not commute* with $\mathcal{N}_cM(-)$, for $i=0,1$.

Some Examples.

(a) $\mathcal{N}_cM(Ext^1_{\mathbf{Z}}(\mathbf{Z}_n\oplus\mathbf{Z}_n,\mathbf{Z}_n))\cong\mathbf{Z}_n^{(b_2)}\ncong 1\cong Ext^1_{\mathbf{Z}}(\mathbf{Z}_n\oplus\mathbf{Z}_n,\mathcal{N}_cM(\mathbf{Z}_n))$, i.e

$$\mathcal{N}_cM(Ext^1_{\mathbf{Z}}(-,A))\ncong Ext^1_{\mathbf{Z}}(\mathcal{N}_cM(-),A).$$

(b) $\mathcal{N}_cM(Ext^1_{\mathbf{Z}}(\mathbf{Z}_n,\mathbf{Z}_n\oplus\mathbf{Z}_n)\cong\mathbf{Z}_n^{(b_2)}\ncong 1\cong Ext^1_{\mathbf{Z}}(\mathcal{N}_cM(\mathbf{Z}_n),\mathbf{Z}_n\oplus\mathbf{Z}_n)$, i.e

$$\mathcal{N}_cM(Ext^1_{\mathbf{Z}}(A,-))\ncong Ext^1_{\mathbf{Z}}(A,\mathcal{N}_cM(-)).$$

(c) $\mathcal{N}_cM(Tor_1^{\mathbf{Z}}(\mathbf{Z}_n\oplus\mathbf{Z}_n,\mathbf{Z}_n))\cong\mathbf{Z}_n^{(b_2)}\ncong 1\cong Tor_1^{\mathbf{Z}}(\mathbf{Z}_n\oplus\mathbf{Z}_n,\mathcal{N}_cM(\mathbf{Z}_n))$, i.e

$$\mathcal{N}_cM(Tor_1^{\mathbf{Z}}(-,A))\ncong Tor_1^{\mathbf{Z}}(\mathcal{N}_cM(-),A).$$

(d) $\mathcal{N}_cM((\mathbf{Z}_n\oplus\mathbf{Z}_n\otimes\mathbf{Z}_n))\cong\mathbf{Z}_n^{(b_2)}\ncong(\mathbf{Z}_n\oplus\mathbf{Z}_n\otimes\mathcal{N}_cM(\mathbf{Z}_n))$, i.e

$$\mathcal{N}_cM(A\otimes -))\ncong(A\otimes\mathcal{N}_cM(-)).$$

(e) $\mathcal{N}_cM(Hom(\mathbf{Z}_n\oplus\mathbf{Z}_n,\mathbf{Z}_n)\cong\mathbf{Z}_n^{(b_2)}\ncong 1\cong Hom(\mathbf{Z}_n\oplus\mathbf{Z}_n,\mathcal{N}_cM(\mathbf{Z}_n))$, i.e.

$$\mathcal{N}_cM(Hom(A,-))\ncong Hom(A,\mathcal{N}_cM(-)).$$

(*f*) $\mathcal{N}_cM(Hom(\mathbf{Z}_{14} \oplus \mathbf{Z}_2, \mathbf{Z}_6 \oplus \mathbf{Z}_3)) \cong \mathbf{Z}_2^{(b_2)} \ncong 1 \cong Hom(\mathbf{Z}_14 \oplus \mathbf{Z}_2, \mathbf{Z}_3^{(b_2)} \cong Hom(\mathbf{Z}_14 \oplus \mathbf{Z}_2, \mathcal{N}_cM(\mathbf{Z}_6 \oplus \mathbf{Z}_3))$, i.e.

$$\mathcal{N}_cM(Hom(A, -)) \ncong Hom(A, \mathcal{N}_cM(-)).$$

(*g*) $\mathcal{N}_cM(Hom(\mathbf{Z}_6 \oplus \mathbf{Z}_2, \mathbf{Z}_9 \oplus \mathbf{Z}_3)) \cong \mathbf{Z}_3^{(b_2)} \ncong 1 \cong Hom(\mathbf{Z}_2^{(b_2)}, \mathbf{Z}_9 \oplus \mathbf{Z}_3)) \cong Hom(\mathcal{N}_cM(\mathbf{Z}_6 \oplus \mathbf{Z}_2), \mathbf{Z}_9 \oplus \mathbf{Z}_3)$, i.e

$$\mathcal{N}_cM(Hom(-, A)) \ncong Hom(\mathcal{N}_cM(-), A).$$

(*h*) $M(Hom(D, \mathbf{Z}_m)) \ncong Hom(M(D), \mathbf{Z}_m)$, and $M(D \otimes \mathbf{Z}_m) \ncong M(D) \otimes \mathbf{Z}_m$, when D is not abelian; Because one can see that $Hom(S_n, \mathbf{Z}_2) \cong \mathbf{Z}_2$, for $n \geq 2$. Also we know that $M(S_n) \cong \mathbf{Z}_2$, for each $n \geq 4$, see [7, theorem 2.12.3]. Now

$$1 \cong M(Hom(S_n, \mathbf{Z}_2)) \ncong Hom(M(S_n), \mathbf{Z}_2) \cong \mathbf{Z}_2,$$

Moreover $S_n \otimes \mathbf{Z}_2 \cong S_n/S_n' \otimes \mathbf{Z}_2 \cong \mathbf{Z}_2 \otimes \mathbf{Z}_2 \cong \mathbf{Z}_2$. Then

$$1 \cong M(S_n \otimes \mathbf{Z}_2)) \ncong M(S_n) \otimes \mathbf{Z}_2 \cong \mathbf{Z}_2.$$

The functor $\mathcal{S} = A \otimes -$, where A is a non-cyclic group *does not commute* with the fuctor $\mathcal{N}_cM(-)$. Put $A = \mathbf{Z}_{n_1} \oplus \mathbf{Z}_{n_2}, G = \mathbf{Z}_n$, where $n_2|n_1$. Then $A \otimes G \cong \mathbf{Z}_{m_1} \oplus \mathbf{Z}_{m_2}$, where $m_i = (n, n_i)$, for $i = 1, 2$. Clearly $m_2|m_1$, so by Theorem 3.1 we have $\mathcal{N}_cM(A \otimes G) \cong \mathbf{Z}_{m_2}^{(b_2)}$. On the other hand, we have $A \otimes \mathcal{N}_cM(G) \cong A \otimes 1 = 1$. Hence $\mathcal{N}_cM(A \otimes G) \ncong A \otimes \mathcal{N}_cM(G)$.

We should also point out that the Theorem 3.1 shows that the c-nilpotent multiplier functor, $\mathcal{N}_cM(-)$, *does not preserve* the tensor product, for

$$\mathcal{N}_cM(\mathbf{Z}_m \otimes G_{ab}) \ncong \mathcal{N}_cM(\mathbf{Z}_m) \otimes \mathcal{N}_cM(G_{ab}) = 1\ . \ \square$$

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